Review for Midterm II[1]

Assigned: March 30, 2021

Multivariable Calculus MATH 53
with Professor Stankova

Contents

1 Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, and to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

1. a *function of several variables*? The *domain* and *range* of such functions?
2. the *graph* of a two-variable function?
3. a *level curve* and a *contour map* of a two-variable function?
   a *level surface* of a three-variable function?
4. the *limit* (at a point) of a multi-variable function?
5. a *continuous* function of several variables? A *discontinuous* function?
   How to determine that a function is discontinuous?
6. the *operations* (addition, subtraction, multiplication, division, composition) on continuituous functions? When is continuity preserved?
7. a *partial derivative*? *Higher order* partial derivatives?
8. a *partial differential equation (PDE)*? The *Laplace equation*? What are some solutions?
9. a *differentiable function*? The *linear approximation* of a differentiable function? A *tangent plane* to a two-variable differentiable function?
10. a *tree diagram* of a function? A *branch* of such a tree? What are they used for?

11. *implicit differentiation*? When do we have to use it?
12. the *Wave equation*? *Traveling wave* to the right/left? What are some/all solutions?
13. *directional derivatives*? How are they related to partial derivatives?
14. *the gradient vector of a function*? What is its relation to maximal and minimal rates of change? What is its geometric interpretation?
15. the *tangent line* to the level curve of a two-variable function and the *tangent plane* to the level surface of a three-variable function? a *normal* to the level curve at point P?
16. a *critical point*? A *local* maximum/minimum? A *saddle point*?
17. a *global maximum/minimum*?
18. *interior*, *exterior*, and *boundary* points of a set?
19. a *closed set*? A *bounded set*? Why are closed bounded sets "nice" for optimization problems?
20. a *constrained* optimization problem?
21. a *Lagrange multiplier*?
22. the *AM-GM inequality*?
23. the *double integral* of a function of two variables over a rectangle? Over any region in the plane? What is its geometric interpretation?
24. a *Riemann sum* for a double integral over a rectangle?
25. the *average value* of a function over a 2d region?
26. *partial* integration?
27. *iterated* integral? How many such iterated integrals are there?
28. *cross-section* of a solid? *Cavalieri's principle*?
29. a *type I* region in the plane? A *type II*? What are they used for?
30. *switching* the *order* of integration?

2 Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

1. **"Contrapositive theorem"**: If $f(x, y)$ has different limits along two different paths approaching (a, b) (or one of them does not exist), then the limit of f at (a, b) does not exist.
2. **Continuity theorem:** Functions defined by algebraic expressions involving addition, multiplication, division, exponentiation, logs and (inverse) trig functions, and composition of such, are continuous where they are *well-defined* (i.e., where denominators are not zero, expressions inside square roots are non-negative, etc.).
3. **Clairaut's theorem:** If $f(x, y)$ has **continuous** mixed partial derivatives f_{xy} and f_{yx} on a disc $D_{(a,b)}$ inside the domain D_f, then $f_{xy} = f_{yx}$ on this disc.
4. **Sufficient condition for differentiability** (using partial derivatives): If $f_x(x, y)$ and $f_y(x, y)$ exist near (a, b) and are **continuous** at (a, b), then $f(x, y)$ is differentiable at (a, b).
5. The **linearization** of $f(x, y)$ at (x_0, y_0) is given by:
$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$
The **tangent plane** to the graph of $f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is given by:
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$
The **linear approximation** of $f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is given by:
$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

6. **The Chain Rule** for a variety of situations, paraphrased with the gradient vector.
7. **Implicit function theorem:** Suppose that x and y satisfy some equation $F(x, y) = 0$. If $\frac{\partial F}{\partial y}(a, b) \neq 0$, then, *in theory*, the equation can be solved for value of x close to a, giving values of y close to b. *In practice*, however, it might not be possible to find a expression for $y(x)$; we can still find $\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}$, using implicit differentiation.

   Similarly, if $F(x, y, z) = 0$ and F_x, F_y, F_z exist, with $F_y \neq 0$, then $\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}$ and $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$.
8. **Solutions to the wave equation:** All solutions $u(t, x)$ to the Wave equation $u_{tt} = c^2 u_{xx}$ are of the form $u(t, x) = g(x - ct) + h(x + ct)$, for some single-variable, twice-differentiable functions $g(y)$ and $h(y)$.
9. **Formula for directional derivatives:** $D_{\vec{u}} f(x, y) = \nabla f(x, y) \circ \vec{u}$ for any *unit* vector $\vec{u} \in \mathbb{R}^2$. This shows that the gradient $\nabla f(x, y)$ points in the direction of the fastest growth of f at (x, y) and the rate of this fastest growth is $|\nabla f(x, y)|$.
10. **Formula for tangent lines/planes of level sets:**
    - If $f(x, y) = c$ is a level curve of a differentiable function $f(x, y)$ and $P = (x_0, y_0)$ is a point on this level curve, then $\nabla f(P) \perp$ the level curve at P.
      Hence, the tangent line to the level curve at P is given by:
      $$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) = 0.$$
    - If $f(x, y, z)$ is differentiable at $P = (x_0, y_0, z_0)$, then its tangent plane at P has normal vector $\nabla f(x_0, y_0, z_0)$; i.e., it is given by:
      $$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0.$$
11. **Necessary Condition for Local Extrema:** If $f(x, y)$ is differentiable and has a local extremum at (x_0, y_0), then $\nabla f(x_0, y_0) = \langle 0, 0 \rangle$; i.e., $f(x, y)$ has a critical point at P.
12. **2nd Derivative Test:** If $f(x, y)$ has a critical point at (a, b), and all 4 2nd-order partial derivatives are continuous nearby (a, b), set $D = \left(f_{xx} f_{yy} - f_{xy}^2\right)\big|_{(a,b)}$.
    (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
    (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
    (c) If $D < 0$, then $f(a, b)$ is a saddle point.
    (d) If $D = 0$, the test fails to reach a conclusion. We need another test!
13. **Extreme Value Theorem:** If $f(x, y)$ is continuous on a **closed** and **bounded** domain D_f, then f has a global minimum and a global maximum (somewhere on D_f).
14. **Nice Domain Method** If $f(x, y)$ has continuous partial derivatives f_x and f_y on a closed and bounded domain D_f in $\mathbb{R}^2$, then the global extrema of f on D_f are among the two sources:
    (a) critical points: $\nabla f(x, y) = \vec{0}$ for some $(x, y) \in D_f$.
    (b) extrema of f along the boundary ∂D_f.
15. **Lagrange Multipliers:** If $f(x, y)$ has a global extremum along the constraint curve $g(x, y) = k$, and both functions are differentiable, with $\nabla g \neq \vec{0}$, then this global extremum is obtained at one of the solutions (x_0, y_0) of the system, where λ is called a *Lagrange multiplier*:
    $$\left|\begin{array}{rcl} f_x(x_0, y_0) & = & \lambda g_x(x_0, y_0); \\ f_y(x_0, y_0) & = & \lambda g_y(x_0, y_0); \\ g(x_0, y_0) & = & k. \end{array}\right.$$
16. **Fubini's Theorem:** If $f(x, y)$ is continuous on a rectangle $R = [a, b] \times [c, d]$ then $\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$.
17. **Type I Region:** If D is of type I region in $\mathbb{R}^2$; i.e., $a \leq x \leq b$, and for any such fixed x, $g(x) \leq y \leq h(x)$, and $f(x, y)$ is a function defined on D, then
    $$\iint_D f(x, y)\ dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\ dy\ dx.$$

3 Problem Solving Techniques

1. **Proving that a limit does not exist:** To show that a limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist, we could attempt to find two different paths approaching the origin such that $f(x,y)$ has different limits along those paths. Alternatively we can also try to find a path along which the limit does not exist. Some comments/tips:
   - Usually, it will be easy to compute the limits along the axes $\lim\limits_{x\to 0} f(x,0)$ and $\lim\limits_{y\to 0} f(0,y)$, so we should do that first and then search for a path giving a different limit.
   - Typically, the next best thing to try are the diagonals $y = x$ and $y = -x$.
   - If all these still produce the same limit, we can try a more general line $y = mx$ for some parameter m and see if for some m the limit is different from the above.

     Example: We show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ where $f(x,y) = \frac{x^3y-xy^3}{x^4+y^4}$ does not exist:
     First, we see that $f(x,0) = 0$, so the limit along $y = 0$ is zero. Taking the limit along $x = 0$ and $y = x$ also gives 0, so this doesn't help. Second, we try $y = mx$ and see that
     $$f(x,mx) = \frac{mx^4-m^3x^4}{x^4+m^4x^4} = \frac{m(1-m^2)}{1+m^4}.$$
     Evidently, $\lim\limits_{x\to 0} f(x,mx) = \frac{m(1-m^2)}{1+m^4}$, so picking, say, $m = \frac{1}{2}$ gives a limit $\neq 0$. $\square$

   - If $y = mx$ still doesn't work, we can try to find the limits along $y = x^\alpha$ or even $y = mx^\alpha$, where α is some parameter. It might be beneficial to guess a value of α that makes $f(x, x^\alpha)$ particularly simple.

     Example: We show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ where $f(x,y) = \frac{yx^2}{x^6+y^2}$ does not exist.
     We quickly see that $f(x,0) = 0$, so we need to find a path along which the limit is not 0. If we try $y = mx$, we will still get 0 as the limit, so we try $y = x^\alpha$. We observe that:
     $$f(x,x^\alpha) = \frac{x^{2+\alpha}}{x^6+x^{2\alpha}}.$$
     Our denominator becomes quite simple if $2\alpha = 6$, so we might try $\alpha = 3$. Indeed, $f(x,x^3) = \frac{1}{2x}$ does not even have a limit as $x \to 0$. (So looking back, we wouldn't even have had to consider any other path before.) $\square$

   - Technically, our path could be any parametric curve; e.g. $x(t) = e^{-t}\cos t, y(t) = e^{-t}\sin t$ as $t \to \infty$. However, it is mostly sufficient to consider paths of the form $y = g(x), x \to 0$ or $x = g(y), y \to 0$ for some function $g(x)$ such that $\lim\limits_{x\to 0} g(x) = 0$.
2. The **Chain Rule:** Let $f(x,y)$ be a function of two variables x and y, which, in turn, are functions of another variable t. Then
   $$\frac{d}{dt} f(x(t),y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$
   If x and y instead are functions of two variables s,t, we have:
   $$\frac{d}{dt} f(x(t,s),y(t,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{d}{ds} f(x(t,s),y(t,s)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$
   This should be thought of as a chain of events: In the latter case, we want to know how a (small) change Δs in s affects $f(x(t,s),y(t,s))$. This happens in two different ways:
   (a) A change Δs in s causes a change $\Delta x = \frac{\partial x}{\partial s}\Delta s$ in x, which, in turn, causes a change $\Delta_1 f = \frac{\partial f}{\partial x}\Delta x$ in $f(x,y)$.
   (b) At the same time, the change in s causes a change $\Delta y = \frac{\partial y}{\partial s}\Delta s$ in y and, thus, another change $\Delta_2 f = \frac{\partial f}{\partial y}\Delta y$ in $f(x,y)$.

   Therefore, the total change in $f(x,y)$ is $\Delta f = \Delta_1 f + \Delta_2 f = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\right)\Delta s$.

3. **Implicit Differentiation:** Let x, y and z satisfy some equation $F(x, y, z) = 0$. Suppose (see the Implicit Function Theorem earlier) that we could, in theory, solve this equation for x, obtaining x in terms of y and z. Then we could take the partial derivatives $\partial x/\partial y$ and $\partial x/\partial z$. There is a shortcut for computing this, without actually solving for x:
$$\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x};$$
$$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x}.$$
4. **Finding local mimima/maxima** of a function $f(x, y)$. This method also works when we are asked to find and classify the critical points.
   1. Compute $\nabla f(x, y)$ and find the solutions of $\nabla f(x, y) = \vec{0}$: these are the critical points.
   2. Use the **2nd derivative test** for every critical point to classify it; namely, compute $D = f_{xx}f_{yy} - f_{xy}^2$ at the critical point. If:
      - $D > 0$ and $f_{xx} > 0$, then $f(x, y)$ is a local minimum.
      - $D > 0$ and $f_{xx} < 0$, then $f(x, y)$ is a local maximum.
      - $D > 0$ and $f_{xx} = 0$, go back and redo your computations of f_{xx} and D because this never happens!
      - $D < 0$, then $f(x, y)$ is a saddle point, so it is neither a minimum nor a maximum.
      - $D = 0$, then we can't say anything about this point using the 2nd derivative test.
5. **Finding global minima/maxima on closed and bounded domains** for a function $f(x, y)$ defined on D_f. This is done in three steps:
   1. Find the critical points of f in the interior of D_f by solving $\nabla f(x, y) = \langle 0, 0\rangle$. (There might be solutions that don't lie in D_f but we ignore them.) Classifying the criticial with the second derivative test (or otherwise) is not necessary here.
   2. Find the extrema of f along the boundary ∂D_f. Sometimes this requires breaking the boundary up into pieces where we can either reduce $f|_{\partial D_f}$ to a single-variable calculus problem (e.g., parametrize lines or circles and plug them into f) or we can use the method of Lagrange Multipliers.
   3. Compute the value of f on all the thus-found critical points in the interior and extrema along the boundary, and pick the maxima and minima among them.
6. **Lagrange Multipliers:** We want to maximize/minimize a function $f(x, y)$ on a curve described by an equation $g(x, y) = k$.
   1. Compute ∇f and ∇g.
   2. Solve the system of equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = k$ for x, y, and λ (we only need the solutions for x and y).
   3. Compute the value of $f(x, y)$ for every solution and decide which one(s) are the maxima/minima.

4 Problems for Review

The exam will be based on Homework, Lecture, Section and Quiz problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 4 representative problems from various HW assignments, give yourself 40 minutes, and then compare your solutions to the VW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

4.1 Limits and Continuity

1. True/False practice:
   (a) If f is a function whose domain contains points arbitrarily close to $(2,3)$, then $\lim_{(x,y)\to(2,3)} f(x,y) = (2,3)$.
   (b) If $\lim_{(x,y)\to(a,b)}$ exists, the function $f(x,y)$ must be defined or continuous at (a,b).
   (c) The function $f(x,y) = x - y + 1$ is not continuous at the point $(0,1)$.
   (d) To show that the limit at a point (a,b) exists, it suffices to find two paths to the point (a,b) where the limits of $f(x,y)$ agree.
   (e) To prove that the limit of $f(x,y)$ at $(x,y) \to (a,b)$ does not exist, we have to prove that the limits along at least 3 different paths are different.
   (f) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along any line through (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.
   (g) The (ϵ, δ)-definition of limits and continuity can be extended to functions of 3 and more variables.
   (h) If f and g are two continuous functions in $\mathbb{R}$, $(a,b) \in \mathbb{R}^2$ and $\lim_{x\to a} f(x) = L_1$ and $\lim_{y\to b} g(y) = L_2$ then $\lim_{(x,y)\to(a,b)} f(x)g(y) = L_1 L_2$.
2. Show that the limit does not exist.
   (a) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+xy^3}$;
   (b) $\lim_{(x,y)\to(1,0)} \frac{x+y^2}{(x-1)^3+y^3}$.
3. Discuss continuity of the following functions:
   (a) $\frac{2x^4y}{x^8+y^2}$;
   (b) $\frac{x^2\cos^2 x}{x^2+(y-1)^2}$.
4. Find the domain of the following functions and prove that they are continuous there:
   (a) $f(x,y) = x^2 + 2xy + e^x - \cos y - 2$;
   (b) $f(x,y) = \frac{xy^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. (More than one solution?!)

4.2 Partial Derivatives

1. True/False practice:
   (a) Clairaut's Theorem says that $f_{xy} = f_{yx}$.
   (b) $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$.
   (c) Any second partial derivative of a sum is the sum of the corresponding second partial derivatives, assuming all these derivatives exist:
   $$\frac{\partial^2(f_1+f_2)}{\partial w^2} = \frac{\partial^2 f_1}{\partial w^2} + \frac{\partial^2 f_2}{\partial w^2}.$$
2. Implicit differentiation:
   (a) Find $\frac{\partial y}{\partial x}$ if $x^2 + 2xy^2 + z^3 + xyz + y = 2$.
   (b) Find $\frac{\partial x}{\partial y}$ for the above equation.
   (c) Find $\frac{\partial y}{\partial x}$ for $e^y \sin x = x + xy$.
3. Higher order computation:
   (a) Prove that $c(x,t) = \frac{1}{\sqrt{Dt}} e^{-\frac{x^2}{4}Dt}$ is a solution of the diffusion equation $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$.
   (b) *(Computation avid?)* Find $\frac{\partial^2 c}{\partial t^2}$ for the above function $c(x,t)$.
4. The *Van der Waals* equation equation of state for a gas is $\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$, where p is the pressure, V the volume, T the temperature, n the amount of moles in the gas, and

R, a, b are positive constants. We can always assume that $V > nb$.

(a) Calculate $\frac{\partial T}{\partial p}$ and $\frac{\partial T}{\partial V}$.

(b) Give the linear approximation of T for a small increase of p and V.

(c) Find the *critical point* of a Van der Waals gas (p_c, V_c) at which $\frac{\partial p}{\partial V} = \frac{\partial^2 p}{\partial V^2} = 0$. (This is a challenging computation!)

4.3 Tangent Planes and Linear Approximations

1. True/False practice:
   (a) The linear approximation $L_{(a,b)}(x, y)$ of a function $f(x, y)$ is always a good way to approximate the function around (a, b).
2. Prove that if f is a function of two variables that is differentiable at (a, b), then f is continuous at (a, b). (Hint: go back to the definitions!)
3. Find the equation of the tangent plane to $z = f(x, y) = x^2 \cos(\pi y) - \frac{6}{xy^2}$ at $(2, -1)$.
4. Find the linear approximation to $z = \cos(\sin y - x)$ at $(-2, 0)$.

4.4 Chain Rule

1. True/False practice:
   (a) For $u = f(x, y)$, where $x = x(r, s, t), y = y(r, s, t)$, we can find $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$. (Assume all functions are differentiable.)
2. A function f is called homogeneous of degree n if it satisfies the equation

$$f(tx, ty) = t^n f(x, y) \tag{4.1}$$

   for all t, where n is a positive integer and f has continuous second order derivatives.
   (a) Verify that $f(x, y) = x^2 y + 2xy^2 + 5y^3$ is homogeneous of degree 3.
   (b) Show that if f is homogeneous of degree n then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$. (Hint: use the Chain Rule to differentiate $f(tx, ty)$ with respect to t.)
   (c) If f is homogeneous of degree n, show that $f_x(tx, ty) = t^{n-1} f_x(x, y)$ for $t > 0$.
3. Let $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f be differentiable. Prove that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.
4. Let $f(u - v^2, u^3 + v)$ be differentiable, and so be its derivatives. Find $\frac{\partial^2 f}{\partial u \partial v}$.

4.5 Gradient Vector

1. True/False practice:
   (a) The gradient vector $\nabla f(a, b)$ for a two-variable function $z = f(x, y)$ lives in 3d space and is perpendicular to the tangent plane of the graph at $(a, b, f(a, b))$.
   (b) $D_{\vec{\imath}+\vec{\jmath}} f(x, y) = f_x \vec{\imath} + f_y \vec{\jmath}$.
   (c) The gradient of a function is always orthogonal to the direction of maximum change of the function.
2. Find the maximum rate of change and its direction for $f(x, y) = \sqrt{x^2 + y^2}$ at $(-1, 1)$.

4.6 Extrema

1. True/False practice:
   (a) If $f(x, y)$ has two local maxima, then it must have a local minimum too.
   (b) The normal vector to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$\langle f_x(a, b), f_y(a, b), -1 \rangle.$$

(c) To find the maximum of $f(x,y)$, one simply needs to find the points (a,b) at which $f_x(a,b)=0$ and $f_y(a,b)=0$.
(d) Suppose the 2nd partial derivatives of D is continuous on a disk near (a,b). Then using the 2nd derivative test, if the determinant $D>0$ and $f_{yy}(a,b)>0$, we cannot determine if this is a local minimum or maximum because we do not know the sign of $f_{xx}(a,b)$.
(e) The normal vector to the surface $z=f(x,y)$ is three-dimensional, while the normal vector to the level curve of $z=f(x,y)$ is two-dimensional.
(f) The graph of $f(x,y)=x^2-xy-y^2$ has a saddle point.

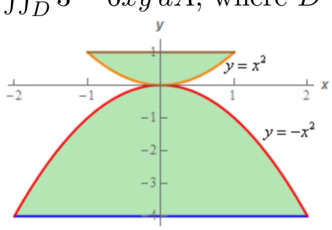
2. Find a classify all the critical points of the following functions:
   (a) $f(x,y)=7x-8y+2xy-x^2+y^2$;
   (b) $f(x,y)=(3x+3x^3)(y^2+2y)$;
   (c) $f(x,y)=(y-2)x^2-y^2$;
   (d) $f(x,y)=xye^{x^2+y^2}$.

4.7 Lagrange Multipliers

1. True/False practice:
   (a) The method of Lagrange multipliers gives us an efficient method to find the intersection between the plane $z=2x-y+3$ and the ellipsoid $x^2+y^2+z^2=1$.
   (b) To find the extrema of a function via MLM, we must find (among other things) the value of the corresponding Lagrange multiplier.
2. Consider $f(x,y)=xy$ and $x^2-y=12$. We assume $y\le 0$.
   (a) Why do we need $y\le 0$ here?
   (b) Find the extreme values of f subject to the above constraints.
3. Consider $f(x,y,z)=xyz$ and $g(x,y,z)=x+y^2+9z^2=4$. We assume $x\ge 0$.
   (a) Why do we need $x\ge 0$ here?
   (b) Find the extreme values of f subject to the above constraints.

4.8 Integrals

1. True/False practice:
   (a) For a continuous function f, the value $\iint_R f(x,y)\,dA$ can be viewed as a volume.
   (b) For a continuous function f, $\int_0^1\int_0^y f(x,y)\,dx\,dy=\int_0^1\int_0^x f(x,y)\,dy\,dx$.
   (c) $\iint_D f(x,y)\,dA=\iint_{D_1} f(x,y)\,dA+\iint_{D_2} f(x,y)\,dA$ if $D=D_1\cup D_2$.
2. Let $I=\iint_D 5x^3\cos(y^3)\,dA$ where D is the region bounded by $y=2, y=x^2/4$ and $x\ge 0$.
   (a) Make a quick sketch of the area of interest.
   (b) Evaluate I on D.
3. Find the volume enclosed under the plane $3x+2y-z=0$ and above the region between by the parabolas $y=x^2$ and $x=y^2$.
4. Evaluate the following integrals over the following regions:
   (a) $\iint_D x(y-1)\,dA$, where D is bounded by $y=1-x^2$ and $y=x^2-3$.
   (b) $\iint_D 3-6xy\,dA$, where D is shown below.



4.9 Old Math 53 Exam Problems

1. Do the following limits exist? If a limit exists, explain why and find the limit. If a limit does not exist, explain why not. (Hint: The (ϵ, δ)-definition doesn't have to be mentioned here.)
   (a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y}{\sqrt{x^2+y^2}}$.
   (b) $\lim\limits_{(x,y)\to(\pi,\pi)} \frac{\cos(x+y)}{x+y}$.
2. A mountain lion runs on a mountain whose height above the point (x, y) is $z = x^2 + \sin^2(xy)$.
   (a) In which direction(s) should the mountain lion run from point $(1, 0, 1)$ so that the height is increasing at the fastest possible rate? What is this fastest rate?
   (b) In which direction(s) should the mountain lion run from point $(1, 0, 1)$ so that the height is increasing at **half** of the fastest possible rate?
3. Find the absolute maximum and minimum values attained by the function
$$f(x, y) = xy + 12(x + y) - (x + y)^2$$
   on the triangle between lines $x = 0$, $y = 0$, and $x + y = 10$.
4. It is known that an ellipsoid given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (with $a, b, c > 0$) has volume $\frac{4}{3}\pi abc$. Consider all such ellipsoids that pass through point $(3, 2, 1)$, and among them, find the ellipsoid enclosing the **least** volume.

5 No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No calculators are allowed on the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.) DSP students with special writing or related disability should consult with the instructor regarding their cheat sheets.
- You must submit your cheat sheet on Gradescope by 11AM **before** the exam.
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**